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A MODEL OF THE TWO-VELOCITY MOTION OF A BAROTROPIC TWO-PHASE MEDIUM†

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A model of the two-velocity motion of a barotropic two-phase medium is obtained in the form of a system of parabolic equations. The results of a numerical implementation of the model, as applied to the discharge of effervescing water, are in good agreement with experiment and with calculated data obtained elsewhere.

It is well known that the fundamental system of equations for the two-fluid model of two-phase flow is hyperbolic. Different physical assumptions lead to the loss of the hyperbolic form of the system. This question has been dealt with in the number of papers [1–5] in which methods for compensating for the non-hyperbolic form of the equations are considered which lead to a significant contraction of its domain. As a rule, it is not possible to achieve complete elimination of the non-hyperbolic form successfully. A detailed analysis of this problem is presented in [4]. The various algorithmic procedures, which suppress the development of instability in the solution of a non-hyperbolic system, can lead to inestimable numerical diffusion. The source of the error lies in the hypothesis regarding the equality of the pressures of the phases and it therefore appears natural to turn to a model with unequal pressures [5]. However, this model cannot be implemented in practice due to the lack of reliable information for constructing the system of closed relationships for determining the coefficients and the right-hand sides of the system of basic equations for this model.

We shall demonstrate the possibility of obtaining a model of a two-phase barotropic flow by changing the type of equations of the initial system when it ceases to be hyperbolic. For this purpose, we use the procedure of the stochastic approximation of a fluctuating parameter and subsequent averaging of the equations over its realizations.

1. The generally accepted operators for averaging the equations of two-phase flows over a number of realizations, space and time [1] are trivial in the sense that the statistical properties of the parameters which are employed in these operators only determine the means [6].

We shall consider an approach which makes use of the non-trivial statistical properties of the parameters of two-phase media. Bearing in mind that space–time fluctuations may exist in flows, it is possible to use a statistical-averaging procedure which takes account of the stochastic nature of the fluctuating parameters. We know (see [7], for example) that, in many physical problems, processes involving a change in the parameters with time can be treated in the approximation of delta-correlated random processes. In particular, when applied to vapour–liquid flows, this approximation of the fluctuating parameters has a quite explicit physical nature: the spontaneous processes of the formation of vapour bubbles and their destruction, and the formation of films and slugs can be treated as jumps in the statistical mean values of

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the parameters for the delta-correlated process under consideration. In the case of values of flow parameters which are distributed in time as the result of the simultaneous action of a set of factors, it may be an acceptable approximation to assume that the fluctuations have a Gaussian form. This approximation is extensively used, for example, when investigating turbulent flows [8–10].

The matrix form of the system of initial hyperbolic equations in the approximation of barotropic phases is [1]

$$\upsilon_{it} + \upsilon_i \upsilon_{ix} + \rho_i^{-1} p_x = \Pi_i \tag{1.1}$$

$$p_t + A_i B_i + B_i C_i p_x + B_i \Delta\upsilon \varphi_{2x} = \Pi_3 \tag{1.2}$$

$$\varphi_{2t} + \varphi_2 (1 - D_i)\upsilon_{2x} - D_i (1 - \varphi_2)\upsilon_{1x} + \tag{1.3}$$

$$+\rho_2^{-1} a_2^{-2} (\upsilon_2 - B_i C_i)\varphi_i p_x + (\upsilon_2 - D_i \Delta\upsilon)\varphi_{2x} = \Pi_4$$

$$h_{it} + \rho_i^{-1} A_i B_i - \rho_i^{-1}(\upsilon_i - B_i C_i) p_x + \rho_i^{-1} B_i \Delta\upsilon \varphi_{2x} + \upsilon_i h_{ix} = \Pi_{i+4} \tag{1.4}$$

where φ, ρ, p, h and υ are, respectively, the volume concentration, density, pressure, enthalpy and velocity of the phases, a is the velocity of propagation of acoustic perturbations in a phase, x is a coordinate, t is the time $i = 1, 2$ (1 refers to the liquid phase and 2 refers to the gas phase), Π are the right-hand sides of the equations, and $\varphi_2 = 1 - \varphi_1$. The following notation was adopted (the summation is over the index i)

$$\Sigma \varphi_i \upsilon_{ix} = A_i, \quad \Sigma \langle \varphi_i \rangle \langle \upsilon_i \rangle = \langle A_i \rangle$$

$$\left(\Sigma \frac{\varphi_i}{\rho_i a_i^2}\right)^{-1} = B_i, \quad \Sigma \frac{\varphi_i \upsilon_i}{\rho_i a_i^2} = C_i$$

$$\Sigma \frac{\langle \varphi_i \rangle \langle \upsilon_i \rangle}{\rho_i a_i^2} = \langle C_i \rangle, \quad \upsilon_2 - \upsilon_1 = \Delta\upsilon$$

$$\langle \upsilon_2 \rangle - \langle \upsilon_1 \rangle = \langle \Delta\upsilon \rangle, \quad \frac{B_i \varphi_2}{\rho_2 a_2^2} = D_i$$

$$\frac{B_i \langle \varphi_2 \rangle}{\rho_2 a_2^2} = \langle D_i \rangle, \quad \Sigma \frac{\sigma_i^2 \langle \varphi_i \rangle}{\rho_i a_i^2} = F_i$$

$$\Sigma \frac{\rho_i a_i^2}{\langle \varphi_i \rangle} = G_i, \quad \Sigma \frac{\sigma_i^2}{\langle \varphi_i \rangle} = H_i$$

$$\Sigma \frac{\rho_i a_i^2 \sigma_i^2}{\langle \varphi_i \rangle} = K_i, \quad \sigma_2^2 - \sigma_1^2 = \Delta\sigma^2$$

As a consequence of the statistical nature of the processes in two-phase flows, there are fluctuations in the velocities of the liquid and gas phases. We shall represent the velocities of the phases υ_i as random functions which are equal to the sum of a mean and a fluctuating term

$$\upsilon_i = \langle \upsilon_i \rangle + \delta\upsilon_i(x, t)$$

Assuming that the fluctuations in the velocities of the phases can be described as a delta-correlated Gaussian process, we introduce the correlation function

$$\langle \delta\upsilon_i(x_1)\delta\upsilon_i(x_2) \rangle = 2\sigma_i^2 \delta(x_1 - x_2) \tag{1.5}$$

where σ_i^2 is the variance of the velocity υ_i. On averaging the equations of system (1.1)–(1.4) over samples of the random process υ_i and expanding the statistical non-linearities which arise here using the Furutsu–Novikov formula and taking account of the correlation (1.5), we obtain a system of equations for the mean values of the required parameters υ_i, p, φ_i and h_i

$$\langle\upsilon_i\rangle_t+\langle\upsilon_i\rangle\langle\upsilon_i\rangle_x-2\sigma_i^2\langle\upsilon_i\rangle_{xx}+\rho_i^{-1}\langle p\rangle_x=\langle\Pi_i\rangle \tag{1.6}$$

$$\begin{aligned}&\langle p\rangle_t+\langle A_i\rangle B_i+B_i\langle C_i\rangle\langle p\rangle_x-2B_i\langle F_i\rangle\langle p\rangle_{xx}+B_i\langle\Delta\upsilon\rangle\langle\varphi_2\rangle_x-\\&-2B_i[\Delta\sigma^2\langle\varphi_2\rangle_{xx}-H_i(\langle\varphi_2\rangle)^2]=\langle\Pi_3\rangle\end{aligned} \tag{1.7}$$

$$\begin{aligned}&\langle\varphi_2\rangle_t+\langle\varphi_2\rangle(1-\langle D_i\rangle)\langle\upsilon_2\rangle_x-\langle D_i\rangle(1-\langle\varphi_2\rangle)\langle\upsilon_1\rangle_x+\\&+G_i^{-1}\langle\Delta\upsilon\rangle\langle p\rangle_x-G_i^{-1}\Delta\sigma^2\langle p\rangle_{xx}+(\langle\upsilon_2\rangle-\langle D_i\rangle\langle\Delta\upsilon\rangle)\langle\varphi_2\rangle_x-2(\sigma_2^2-\langle D_i\rangle\Delta\sigma^2)\langle\varphi_2\rangle_x+\\&+2G_i^{-1}K_i(\langle\varphi_2\rangle_x)^2=\langle\Pi_4\rangle\end{aligned} \tag{1.8}$$

$$\begin{aligned}&\langle h_i\rangle_t+\rho_i^{-1}\langle A_i\rangle B_i-\rho_i^{-1}(\langle\upsilon_i\rangle-B_i\langle C_i\rangle)\langle p\rangle_x+\\&+\rho_i^{-1}B_i\langle\Delta\upsilon\rangle\langle\varphi_2\rangle_x+\langle\upsilon_i\rangle\langle h_i\rangle_x+2\rho_i^{-1}(\sigma_i^2-B_iF_i)\langle p\rangle_{xx}-\\&-2\rho_i^{-1}B_i[\Delta\sigma^2\langle\varphi_2\rangle_{xx}-H_i((\langle\varphi_2\rangle_x)^2]-2\sigma_i^2\langle h_i\rangle_{xx}=\langle\Pi_{i+4}\rangle\end{aligned} \tag{1.9}$$

Averaging of Eqs (1.1)–(1.4) over the stochastic parameter υ_i therefore generates a parabolic system. In particular, averaging of the total derivative of the velocity over its samples generates the Burgers–Hopf operator for the mean value of the velocity so that Eqs (1.6) are analogous to a system of the Burgers–Hopf type. The occurrence of the factors σ_i^2, which have the dimensions of kinematic viscosity, in the second derivatives is associated with the method of "viscosity" in gas dynamics [11]. It is characteristic that the velocities of the phases, which are factors of the first spatial derivatives of the pressure, are replaced in the second derivatives by the variances σ_i^2. Similarly, the slip $\langle\upsilon_2\rangle-\langle\upsilon_1\rangle$ is replaced by the "slip" of the corresponding variances $\sigma_2^2-\sigma_1^2$.

Considerable difficulties arise when investigating the correctness conditions and the properties of the solutions of boundary-value problems for non-linear parabolic equations of the type (1.6)–(1.9) and, at the present time, simpler systems of two equations with a "viscosity" of the Burgers–Hopf type have been insufficiently studied [11].

2. In order to investigate the possibilities of the proposed model, we shall compare the results of its numerical integration with experiment and with the results obtained using tested models. In this case, it is necessary to supplement system (1.6)–(1.9) with initial and boundary conditions and closing relationships which are determined by the formulation of the actual problem.

For system (1.6)–(1.9), we will formulate the problem [12, 13] of the discharge of an effervescing liquid. A tube of length L and constant cross-section area, closed at both ends by membranes, is filled with homogeneous water which has been underheated to the saturation temperature T_s and has a pressure p_0 and is at a temperature $T_{10}\leqslant T_s(p_0)$. At the instant of time $t=0$, the membranes at one of the ends is fractured and, when $t>0$, the effervescing water discharges into the surrounding medium with a pressure p_∞, where $p_\infty\ll p_0$. The flow is assumed to be adiabatic and frictional forces on the walls of the channel are neglected.

The initial conditions for the homogeneous liquid have the form

$$\begin{aligned}&t=0\colon p(x,0)=p_0,\quad T_1(x,0)=T_{10}\\&\varphi_1(x,0)=1,\quad \varphi_2(x,0)=0\end{aligned} \tag{2.1}$$

The boundary condition at the closed end of the tube is the no-flow condition, and, at the open end, equality of the pressures at the section of the tube and in the surrounding medium

$$x = 0:\ \upsilon = 0;\quad x = L:\ p = p_\infty \tag{2.2}$$

System (1.6)–(1.9) with the boundary conditions (2.1) and (2.2) has to be closed with the equations of state of the phases and the relations for the interphase heat exchange and the intensity of the phase transition.

The intensity of interphase heat exchange under bubbling conditions is determined using a formula [14] which takes account of slip and effects associated with the thermal expansion of the bubbles and convective heat transfer

$$\mathrm{Nu}_{1j} = \mathrm{Nu}_0 + 1{,}13\,\mathrm{Pe}^2\left(\frac{1}{13\,\mathrm{Ja}^{3,3} + \mathrm{Pe}^{1,5}} - \frac{6\,\mathrm{Ja}^{0,63}}{31\,\mathrm{Ja}^{4,3} + \mathrm{Pe}^2}\right)$$

$$\mathrm{Nu}_0 = 3{,}9\mathrm{Ja}\left[1 + \frac{1}{2}\left(\frac{\pi}{6\,\mathrm{Ja}}\right)^{2/3} + \frac{\pi}{6\,\mathrm{Ja}}\right]$$

where Ja and Pe are the Jakob modulus and the Peclet number, $\mathrm{Pe} = d/\upsilon_1 - \upsilon_2/a_1^{-1}$, respectively, where a_1 is the thermal diffusivity of water. The subscript j indicates that a quantity refers to the interphase boundary. The well-known approximation in [12] is used for the diameter of a bubble d.

The thermal fluxes per unit volume in the case of interphase heat exchange are

$$\bar{q}_{ji} = q_{ji}A_j = \alpha_{ji}(T_j - T_i)A_j$$

where A is the area of the interphase surface, referred to unit of volume and α is the heat transfer coefficient. Under bubbling conditions, the vapour phase is close to a state of saturation ($T_2 \approx T_1$) and the thermal flux from the interphase surface to the vapour is therefore small, and it can be assumed that the interphase heat exchange is determined by the thermal flux

$$q_{j1} = \alpha_{j1}(T_s - T_1)A_j$$

In the case of bubbling conditions, the area of the interphase surface can be represented by the relation [15]

$$A_j = 6\varphi d^{-1}$$

We then obtain

$$\bar{q}_{j1} = 6\varphi\lambda_1 d^{-2}\,\mathrm{Nu}_1(T_s - T_1) \tag{2.3}$$

where λ_1 is the thermal conductivity of water.

The intensity of the phase transition Q, when there is no interphase friction and no thermal and mechanical effect of the tube walls, has the form

$$Q = \bar{q}_{j1}r^{-1} \tag{2.4}$$

where r is the heat of the phase transition. The velocity of motion of the interphase boundary, which occurs on the right-hand sides of the equations of system (1.6)–(1.9), is determined using the formula in [16]

$$\upsilon_j = 0.5(\upsilon_1 + \upsilon_2) \tag{2.5}$$

The system of equations (1.6)–(1.9) with the boundary conditions (2.1) and (2.2) and the closure relationships (2.3)–(2.5) was integrated numerically using a Lax–Wendroff difference scheme with an artificial viscosity which takes account of the effects of non-linearities [17]. The condition for this scheme to be stable

$$(|\upsilon|+\bar{a})\Delta t/\Delta x<(1+b^2/4)^{1/2}-b/2 \tag{2.6}$$

is more restrictive than the Courant condition, where $\bar{a}$ is the "frozen" velocity of sound, b is a dimensionless constant of the order of unity and Δx, Δt are the spatial and time steps, respectively. Since the numerical solutions depend on the parameters σ_i^2, it is natural to associate them with the stability condition (2.6) and the necessary condition for the approximation of the parabolic system [11]

$$(\Delta x)^2/(2\Delta t)=\sigma_i^2 \tag{2.7}$$

The indeterminacy in expression (2.7) is removable if the inequality $\sigma_2^2>\sigma_1^2$ is taken into account. Then, for fixed Δx, the value of σ_2^2 on the right-hand side of (2.7) corresponds to a smaller time step, which is essential at the initial stages of the effervescence when the flow has a highly non-equilibrium character. On putting $b=\sigma_2^2/\sigma_1^2$ $(\sigma_1\neq 0)$, we obtain the necessary conditions for the choice of the quantities Δx and Δt, taking account of the values of the parameters σ_i^2. Constraints on the integration time step [12] were also taken account of in the initial stages of the effervescence.

The calculated and experimental pressure and volume vapour content in a fixed cross-section ($x=1.39$ m from the closed end) when a tube of length $L=4.1$ m filled with water at a pressure $p_0=6.9$ MPa was depressurized are shown in Fig. 1. The notation adopted here is: 1—experiment [18], 2—the result from [12] when $T_0=515$ K, $n_0=0.5\times10^9$ m^{-3} (n_0 is the initial number of bubbles), 3—our result when $x=1.39$ m, $p_0=6.9$ MPa, $T_0=515$ K, $\sigma_1^2=20$, and $\sigma_2^2=25$, and 4—our result when $\sigma_1^2=60$ and $\sigma_2^2=75$.

It is seen that, in accordance with the prediction in [13] regarding the effect of the relative motion of the phases, making allowance for this factor produces better agreement between the calculated data 3 and 4 and the experimental data compared with the model in [12]. The values $\sigma_1^2=20$ and $\sigma_2^2=25$ are close to being optimal in the sense of the stability of the scheme and the volume of the calculations: a reduction in the values of σ_i^2 leads to a breakdown of the stability of the scheme with respect to the time step and to the degeneration of the parabolic system. An increase in σ_i^2 is associated with an unjustified reduction in the time step and an increase in the volume of the calculations.

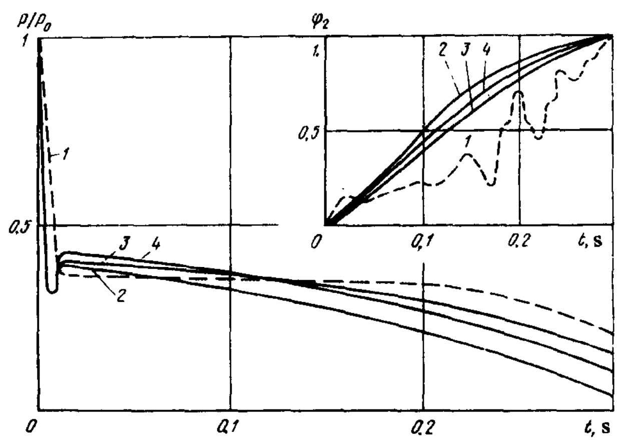

Fig. 1.

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